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REPRESENTATION OF FUZZY ADJACENT MATRICES VIA FUZZY GRAPHS

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ABSTRACT

This paper is an investigation of the fuzzy adjacent matrices on fuzzy graphs. Various operations and fuzzy equivalence relation are calculated in a flexible method. Motivated by the applications of fuzzy matrices the proposed work characterizes some properties composition on standard fuzzy set operations.

Keywords: Fuzzy adjacent matrices, Yager's class, hamacher, bounded, einstein, probabilistic.

INTRODUCTION

The original theory of fuzzy sets was formulated in terms of the operators complement, union and intersection (Zadeh, 1965; Kaufmann, 1975; Zimmermann, 2010). For each of the three set of operations , several different classes of adjacency fuzzy matrix of fuzzy graph possessing appropriate axiomatic properties have subsequently been proposed. Several important properties are shared by all the three operators. Some of the axioms insures that the defined operations on adjacent fuzzy matrix generalizes the classical crisp sets.

Some classes of functions have been proposed whose individual members satisfy the axiomatic requirements of these operations. One of such class is Yager's class (George *et al*., 2009; Hooda and Raich, 2015) which is defined for the fuzzy adjacent matrix by specifying the certain values to the parameters. The parameter determines the strength of union and intersection operations. The paper is organized as follows: In section 2, the basic definitions of fuzzy graph theory was coined along with the fuzzy adjacent matrix. While, in section 3 various properties of fuzzy adjacent matrix are presented. In addition to this a discussion is made on complementary fuzzy graph (Sandeep and Sunitha, 2012). This work presents a perspective on characterizing the fuzzy adjacent matrix of a simple undirected fuzzy graph.

2. PRELIMINARIES

Definition 2.1:
A fuzzy subset of a nonempty set S is a mapping σ: $S\rightarrow[0,1]$. A fuzzy relation on S is a fuzzy subset of SxS.

If μ and ν are fuzzy relations, then $\mu o \nu(u,w)$ = Sup{ $\mu(u,v)\Lambda\ \nu(v, w)$:$v$S}and $\mu^k(u,v)$= Sup{$\mu(u\ ,u_1)\Lambda\nu(u_1\ ,u_2)\Lambda\mu(u_2\ ,u_3)\ \Lambda\ldots\Lambda\mu(u_{k-1}\ ,v)$: u_1 , u_2 ,…. $u_{k-1}\in$S}where 'Λ' stands for minimum.

Definition 2.2:
A fuzzy graph (Rosenfield, 1975) is a pair of functions G: (σ, μ) where $\sigma : V\rightarrow[0,1]$ is a fuzzy subset of non-empty set V and $\mu : V\ xV\rightarrow [0,1]$ is symmetric fuzzy relation on σ such that for all x,y in V the condition $\mu(u,v) \leq \sigma(u)\ \Lambda\ \sigma(v)$ is satisfied for all (u,v) in E.

Definition 2.3:
Let G= (σ,μ) be a fuzzy graph. The degree of a vertex u is $d(u)= \Sigma\mu(u,v), \neq v$.

Definition 2.4:
The complement of a fuzzy graph G:(σ,μ) is a fuzzy graph $\bar{G}$:(σ, $\bar{\mu}$), where $\sigma=\bar{\sigma}$ and $\bar{\mu}(u,v)=\sigma(u)\wedge\sigma(v)-\mu(u,v)$ $\forall$ $u,v\in V$.

Definition 2.5:
The adjacency fuzzy matrix of FG is defined as

$$A_{FG}= \begin{cases} \mu(\mathbf{u},\mathbf{v}) & if\ i,j\ in\ the\ neighbourhood \\ \mathbf{0} & otherwise \end{cases}$$

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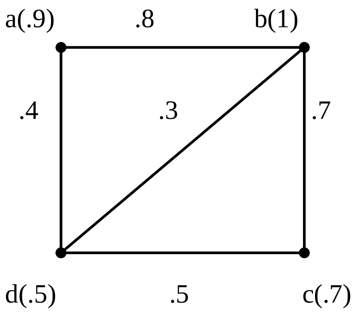

Fuzzy Graph 1.1

The fuzzy adjacent matrix of the above fuzzy graph is given by

$$A_{FG} = \begin{bmatrix} 0 & .8 & 0 & .4 \\ .8 & 0 & .7 & .3 \\ 0 & .7 & 0 & .5 \\ .4 & .3 & .5 & 0 \end{bmatrix}$$

Let us consider two more fuzzy adjacent matrices of same fuzzy graph 1.1

$$B_{FG} = \begin{bmatrix} 0 & .9 & 0 & .8 \\ .9 & 0 & .6 & .3 \\ 0 & .6 & 0 & .6 \\ .8 & .3 & .6 & 0 \end{bmatrix}$$

$$C_{FG} = \begin{bmatrix} 0 & .6 & 0 & .2 \\ .6 & 0 & .5 & .1 \\ 0 & .5 & 0 & .3 \\ .2 & .1 & .3 & 0 \end{bmatrix}$$

Lemma 2.2.1

The fuzzy adjacent matrix of every undirected fuzzy graph G(σ , μ) is symmetric.

Proof:

Let G(σ , μ) be the undirected simple fuzzy graph.

$$\Rightarrow \mu(v_1, v_2) = \mu(v_2, v_1)$$

Let A_{FG} = (a_{ij}) be the fuzzy adjacent matrix of G(σ , μ).

The membership values of A_{FG} of any row and column are same.

$$\Rightarrow A_{FG} = (A_{FG})^t$$

$\Rightarrow A_{FG}$ is symmetric.

Results

- The row(Column) sum of the matrix A_{FG} gives the degree of the corresponding vertex.
- Tr(Tr(A_{FG}) = 0, since G(σ ,μ) is the undirected simple fuzzy graph.
- The fuzzy adjacent matrix of every undirected fuzzy graph G(σ , μ) and its inverse are equal.
- The sum of the membership of each row and column are equal since fuzzy adjacent matrix is symmetric.
- The product of fuzzy adjacent matrix with a crisp(fuzzy) number is again a fuzzy adjacent matrix.

3. COMPLEMENT OF A_{FG}

In this section various properties of A_{FG} under complement operation is discussed (Sandeep and Sunitha, 2012).

$$(A_{FG})' = \begin{bmatrix} 1 & .2 & 1 & .6 \\ .2 & 1 & .3 & .7 \\ 1 & .3 & 1 & .5 \\ .6 & .7 & .5 & 1 \end{bmatrix}$$

Theorem 3.1

The fuzzy graph corresponding to fuzzy adjacent matrix is not complementary and self complementary.

Proof

Let : (σ , μ) be the fuzzy graph corresponding to the complement fuzzy adjacent matrix.

Here μ(u,v) ≤ σ(u) Λ σ(v)

⇒ μ(u,v) ≠ σ(u) Λ σ(v)- μ(u,v) and

μ(u,v) ≠ 1/2[σ(u) Λ σ(v)]

Hence Fuzzy graph is not complementary and self complementary.

Theorem 3.2

The composition of a symmetric fuzzy adjacent matrix and its complement is symmetric.

Proof

Let A_{FG} = (a_{ij}) be the fuzzy adjacent matrix of G(σ , μ).

$\Rightarrow (A_{FG})'$ = 1 - (a_{ij})

By Max-Min composition,

$$\begin{aligned} (A_{FG} \circ A_{FG}{}') &= (\sigma \circ \sigma')(u,v) \\ &= \vee [\sigma(u,v) \wedge \sigma' (u,v)] \\ &= [\vee [\sigma'(u,v) \wedge \sigma(u,v)]]^t \\ &= (\sigma' \circ \sigma)(u,v) \\ &= (A_{FG} \circ A_{FG}{}')^t \end{aligned}$$

Similar result holds for Min-Max composition.

Result

- $((A_{FG})')' = (A_{FG})$

4. OPERATIONS ON FUZZY ADJACENT MATRIX

In this section, the operations of fuzzy sets such as complement , union and intersection are carried out for fuzzy adjacent matrix (Dhar, 2013).

4.1 Characteristics of Fuzzy Adjacent matrix on standard Fuzzy set operators

- Involution $\overline{\overline{A_{FG}}} = A_{FG}$
- Commutative $(A_{FG} \cup B_{FG}) = (B_{FG} \cup A_{FG})$; $(A_{FG} \cap B_{FG}) = (B_{FG} \cap A_{FG})$
- Associative $(A_{FG} \cup B_{FG}) \cup C_{FG} = A_{FG} \cup (B_{FG} \cup C_{FG})$; $(A_{FG} \cap B_{FG}) \cap C_{FG} = A_{FG} \cap (B_{FG} \cap C_{FG})$
- Distributive $A_{FG} \cap (B_{FG} \cup C_{FG}) = (A_{FG} \cap B_{FG}) \cup (A_{FG} \cap C_{FG})$ $A_{FG} \cup (B_{FG} \cap C_{FG}) = (A_{FG} \cup B_{FG}) \cap (A_{FG} \cup C_{FG})$
- Idempotent $(A_{FG} \cup A_{FG}) = A_{FG}$; $(A_{FG} \cap A_{FG}) = A_{FG}$
- Absorption $A_{FG} \cup (A_{FG} \cap B_{FG}) = A_{FG}$; $A_{FG} \cap (A_{FG} \cup B_{FG}) = A_{FG}$
- De Morgan's Law $\overline{(A_{FG} \cap B_{FG})} = \overline{A_{FG}} \cup \overline{B_{FG}}$ $\overline{(A_{FG} \cup B_{FG})} = \overline{A_{FG}} \cap \overline{B_{FG}}$
- Equivalence formula $(\overline{A_{FG}} \cup B_{FG}) \cap (A_{FG} \cup \overline{B_{FG}}) = (\overline{A_{FG}} \cap \overline{B_{FG}}) \cup (A_{FG} \cap B_{FG})$
- Symmetrical Difference Formula $(\overline{A_{FG}} \cap B_{FG}) \cup (A_{FG} \cap \overline{B_{FG}}) = (\overline{A_{FG}} \cup \overline{B_{FG}}) \cap (A_{FG} \cup B_{FG})$

4.2 Fuzzy Union

The fuzzy union of the fuzzy adjacent matrix obeys the following axioms:

Axiom 1: U(Z,Z) = Z ; U(Z,I)=I,U(I,Z) = I ; U(I,I) = I where I is the identity matrix and Z is the zero matrix.

Axiom 2 : $U(A_{FG}, B_{FG}) = U(B_{FG}, A_{FG})$ [Commutative]

Axiom 3 : $U[U(A_{FG}, B_{FG}), C_{FG}] = U[U(A_{FG}, U(B_{FG}, C_{FG})]$ [Associative]

Axiom 4 : $U[A_{FG}, B_{FG}] = A_{FG}$ [Idempotent]

Yager's Union for adjacent Fuzzy Matrix

The Yager's Union ((Hooda and Raich, 2015) for fuzzy adjacent matrix is defined by

$$U_w(A_{FG}, B_{FG}) = Min\ [\ 1, \left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w}]$$

where $w \in (0, \infty)$

For w=1, $U_1(A_{FG}, B_{FG}) = Min\ [\ 1, a_{ij} + b_{ij}]$

For w =2 ,

$$U_2(A_{FG}, B_{FG}) = Min\ [\ 1, \left(a_{ij}{}^{2} + b_{ij}{}^{2}\right)^{1/2}]$$

For w = ∞ $U_\infty(A_{FG}, B_{FG}) = U\ (A_{FG}, B_{FG})$

As w increases, Yager's Union follows decreasing sequence.

Theorem 4.2.1

Let G(σ , μ) be the undirected simple fuzzy graph .Let A_{FG} = (a_{ij}) and B_{FG} = (b_{ij}) be the fuzzy adjacent matrices of same order. If $\cup_w(A_{FG}, B_{FG})$ is the Yager's union function with parameter $w \in (0, \infty)$ then

$$\mathbf{lim}_{w\to\infty} Min\ [\ 1, \left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w} = \text{Max}\ (a_{ij}, b_{ij})$$

Proof

The proof is obvious whenever a_{ij} = 0 and $b_{ij} = 0$ or $a_{ij} = b_{ij}$ because as $w \to \infty$,the limit equals 1 . If $a_{ij} \neq b_{ij}$ and the min equal $\left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w}$ then

$$\lim_{w\to\infty} \left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w} = \text{Max}\ (a_{ij}, b_{ij}).$$

Without loss of generality assume that $a_{ij} < b_{ij}$ and let Y = $\left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w}$

$$\Rightarrow \lim_{w\to\infty} In\ Y = \lim_{w\to\infty} \frac{In\ \left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)}{w}$$

$$= \lim_{w\to\infty} \frac{a_{ij}{}^{w} \boldsymbol{In}\, a_{ij} + b_{ij}{}^{w} In\, b_{ij}}{a_{ij}{}^{w} + b_{ij}{}^{w}}$$

$$= \lim_{w\to\infty} \frac{\left(\frac{a_{ij}}{b_{ij}}\right)^{w} In\ a_{ij} + In\, b_{ij}}{\left(\frac{a_{ij}}{b_{ij}}\right)^{w} + 1}$$

$$= \text{In}\ b_{ij}$$

$$\lim_{w\to\infty} In\ Y = b_{ij} = \text{Max}\ (a_{ij}, b_{ij})$$

It remains to show that the theorem is still valid when min equals 1 .

In this case $\left(a_{ij}{}^{w} + b_{ij}{}^{w}\right)^{1/w} \geq 1$

(i.e) $a_{ij}{}^{w} + b_{ij}{}^{w} \geq 1$ where w$\in$ $(0, \infty)$

When $w \to \infty$, the last inequality holds if $a_{ij} = 1$ or $b_{ij} = 1$ since $a_{ij}, b_{ij} \in [0,1]$

4.2.1 Other Union Operations

- **Probabilistic sum or Algebraic sum**

The probabilistic sum is defined by

$(A_{FG} \hat{+} B_{FG}) = a_{ij} + b_{ij} - a_{ij}\, b_{ij}$

This does not follow commutative, associative and identity law.

- **Bounded Sum or Bold Union**

The bounded sum is defined by

$(A_{FG} \oplus, B_{FG})$ = Min [1, $a_{ij} + b_{ij}$].

This follows commutative, associative and identity law and is identical to Yager's function at w = 1.

- **Drastic Sum**

The drastic sum is defined by

$$(A_{FG} U\ B_{FG}) = \begin{cases} a_{ij} & if\ b_{ij} = 0 \\ b_{ij} & if\, a_{ij} = 0 \\ 1 & otherwise \end{cases}$$

This gives the adjacency matrix of the crisp graph.

- **Hamacher Sum**

The Hamacher sum is given by

$$(A_{FG} U\ B_{FG}) = \frac{a_{ij} + b_{ij} - 2a_{ij}\, b_{ij}}{1 - a_{ij}\, b_{ij}}$$

- **Einstein Sum**

The Einstein sum is given by

$$(A_{FG} U\ B_{FG}) = \frac{a_{ij} + b_{ij}}{1 + a_{ij}\, b_{ij}}.$$

Remark

The Einstein sum of symmetric fuzzy adjacent matrix is symmetric but this is not true for the Hamacher sum.

4.3 Fuzzy Intersection

The fuzzy intersection (Hooda and Raich, 2015) of the fuzzy adjacent matrix obeys the following axioms:

Axiom 1 : ∩ (Z,Z) = Z ; ∩ (Z,I)=I,∩(I,Z) = I ; ∩ (I,I) = I where I is the identity matrix and Z is the zero matrix.

Axiom 2 : ∩ (A_{FG}, B_{FG}) =∩ (B_{FG}, A_{FG}) [Commutative]

Axiom 3 : ∩ [∩ $(A_{FG}, B_{FG}), C_{FG}$] = ∩ [∩ $(A_{FG}$,∩ (B_{FG}, C_{FG})] [Associative]

Axiom 4 : ∩ $[A_{FG}, B_{FG}] = A_{FG}$ [Idempotent]

Yager's Intersection for Fuzzy adjacent matrix

The Yager's intersection for fuzzy adjacent matrix is defined by

$\cap_w\ (A_{FG}, B_{FG}) = 1 - Min\ [1, ((1 - a_{ij})^{w} +$ *bij)w1w]* where w$\in(0,\infty)$

For w=1,

$\cap_1\ (A_{FG}, B_{FG}) = 1 - Min\ [1, 2 - a_{ij} + b_{ij}]$

For w =2 ,

$$\cap_2\ (A_{FG}, B_{FG}) = 1 - Min\ [1, ((1 - a_{ij})^2 + b_{ij})^2)^{1/2}$$

For w = ∞ $\cap_\infty\ (A_{FG}, B_{FG})$ = Min (A_{FG}, B_{FG})= ∩ (A_{FG}, B_{FG})

Theorem 4.3.1

Let G(σ , μ) be the undirected simple fuzzy graph .Let A_{FG} = (a_{ij}) and B_{FG} = (b_{ij}) be the fuzzy adjacent matrices of same order. $\cap_w\ (A_{FG}, B_{FG})$ is the Yager's intersection function with parameter w $\in (0, \infty)$ then

$\lim_{w\to\infty}[1 - Min\ [1, ((1 - a_{ij})^{w} +$ *bij)w1w]]* = Min (*aij,bij)*

Proof

From theorem 4.2.1, M$\lim_{w\to\infty}[1 - Min\ [1, ((1 -$ *aij w+bij)w1w]]* =Min (*1−aij,1−bij)*

$\Rightarrow \cap_\infty\ (A_{FG}, B_{FG})$ =1 - Max $(1 - a_{ij}, 1 - b_{ij})$

Without loss of generality assume that $a_{ij} \leq b_{ij}$

$\Rightarrow 1 - a_{ij} \geq 1 - b_{ij}$

$$\Rightarrow \cap_{\infty} (a_{ij}, b_{ij}) = 1 - (1 - a_{ij}) = a_{ij} = Min(a_{ij}, b_{ij}).$$

Hence the theorem

4.3.1 Other Intersection Operations

- **Probabilistic Product or Algebraic Product**

  The probabilistic product is defined by

  $(A_{FG} . B_{FG}) = a_{ij} \; b_{ij}$.

This follows associative and identity law but does not follows commutative law.

- **Bounded Product**

▪ The bounded sum is defined by

$(A_{FG} \odot B_{FG})$ = Max [0, $a_{ij} + b_{ij} - 1$].

▪ This follows commutative, associative and identity law but not the idempotent law.

▪ Also $(A_{FG} \odot Z) = Z$ and $(A_{FG} \odot \overline{A_{FG}}) = Z$

- **Drastic Product**

The drastic product is defined by

$$(A_{FG} \circledcirc B_{FG}) = \begin{cases} a_{ij} & if \; a_{ij} = 1 \\ b_{ij} & if b_{ij} = 1 \\ 1 & if a_{ij}, b_{ij} < 1 \end{cases}$$

Also $(A_{FG} \circledcirc B_{FG}) = Z$

- **Hamacher Product**

The Hamacher product is given by

$$(A_{FG} \cap B_{FG}) = \frac{a_{ij} \, b_{ij}}{[a_{ij} + b_{ij} - a_{ij} \, b_{ij}]}$$

- **Einstein Product**

The Einstein product is given by

$$(A_{FG} \cap B_{FG}) = \frac{a_{ij} \, b_{ij}}{2 + [a_{ij} + b_{ij} - a_{ij} \, b_{ij}]}$$

- **Hamacher Intersection**

The Hamacher intersection is given by ,

$$(A_{FG} \cap B_{FG}) = \frac{a_{ij} \, b_{ij}}{\gamma + (1 - \gamma)[a_{ij} + b_{ij} - a_{ij} \, b_{ij}]},$$

$\gamma \geq \mathbf{0}$

Remark

In Hamacher intersection , $\gamma = \mathbf{0}$ gives Hamacher product, $\gamma = \mathbf{1}$ gives product, $\gamma = \mathbf{2}$ gives Einstein product and $\gamma = \mathbf{\infty}$ gives drastic product.

Applications

The obtained results can be applied in various areas of engineering, computer science: artificial intelligence, signal processing, pattern recognition, robotics, computer networks, expert systems, clustering and medical diagnosis (Ghanzinoory *et al*., 2010). The fuzzy adjacent matrix can be applied in various decision making such as routing, supplier selection decisions etc (Bondy and Murty, 1976; Timothy, 1997; Frank, 2001; Kwang, 2005).

CONCLUSION

In this paper, operations on fuzzy set theory is extended to the fuzzy adjacent matrix of the fuzzy graph. In this process many properties of fuzzy adjacent matrix is discussed. It has been identified that the zero and identity matrices of crisp set theory are fuzzy matrices.

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